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An algebraic q-deformed form for shape-invariant systems

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Abstract
A quantum deformed theory applicable to all shape-invariant bound-state systems is introduced by defining q-deformed ladder operators. We show that these new ladder operators satisfy new q-deformed commutation relations. In this context we construct an alternative q-deformed model that preserves the shape-invariance property presented by the primary system. q-deformed generalizations of Morse, Scarf and Coulomb potentials are given as examples.

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1. Introduction

Supersymmetric quantum mechanics [1, 2] is generally studied in the context of one-dimensional systems. The partner Hamiltonians

$$\hat{H}_1 = \hbar\Omega \hat{A}^\dagger \hat{A} \qquad \text{and} \qquad \hat{H}_2 = \hbar\Omega \hat{A}\hat{A}^\dagger \tag{1.1}$$

are most readily written in terms of one-dimensional operators

$$\hat{A} \equiv \frac{1}{\sqrt{\hbar\Omega}}\left(W(x) + \frac{\mathrm{i}}{\sqrt{2m}}\hat{p}\right) \qquad \text{and} \qquad \hat{A}^\dagger \equiv \frac{1}{\sqrt{\hbar\Omega}}\left(W(x) - \frac{\mathrm{i}}{\sqrt{2m}}\hat{p}\right) \tag{1.2}$$

where $\hbar\Omega$ is a constant energy scale factor, introduced to permit working with dimensionless quantities, and $W(x)$ is the superpotential. A number of such pairs of Hamiltonians share an integrability condition called shape invariance [3]. The shape-invariance condition has an underlying algebraic structure [4]. Note that the definition in equation (1.1) differs from that given in [4] as in the present paper all operators are defined to be dimensionless.

[4] Now deceased.

The development of quantum groups and quantum algebras motivated great interest in q-deformed algebraic structures [5–7], and in particular in the q-harmonic oscillators. Until now quantum groups have found applications in solid-state physics [8], nuclear physics [9, 10], quantum optics [11] and conformal field theories [12]. Quantum algebras are deformed versions of the usual Lie algebras obtained by introducing a deformation parameter q. In the limit of q going to unity these quantum algebras reduce to the usual Lie algebra. The quantum algebras provide us with a class of symmetries which is richer than the usual class of Lie symmetries; the latter is contained in the former as a special case (when $q \to 1$). Quantum algebras may turn out to be appropriate tools for describing symmetries of physical systems which cannot be described by ordinary Lie algebras.

In this paper we present a quantum-deformed generalization for all shape-invariant systems and show that it is possible to introduce generalized ladder operators for these systems. The paper is organized in the following way: in section 2 we present the algebraic formulation of shape invariance. In section 3 we introduce the fundamental principles of quantum-deformed generalization. In section 4 we introduce the shape-invariant formulation for quantum-deformed systems and work out some examples. Finally, section 5 concludes the paper.

2. Algebraic formulation to shape invariance

The Hamiltonian $\hat{H}_1$ of equation (1.1) is called shape invariant if the condition

$$\hat{A}(a_1)\hat{A}^\dagger(a_1) = \hat{A}^\dagger(a_2)\hat{A}(a_2) + R(a_1) \tag{2.1}$$

is satisfied [3]. In this equation a_1 and a_2 represent parameters of the Hamiltonian. The parameter a_2 is a function of a_1 and the remainder $R(a_1)$ is independent of the dynamical variables such as position and momentum. As is written the condition of equation (2.1) does not require the Hamiltonian to be one dimensional and one does not need to choose the ansatz of equation (1.2). In the cases studied so far the parameters a_1 and a_2 are either related by a translation [4, 13] or a scaling [14–16]. Introducing the similarity transformation that replaces a_1 with a_2 in a given operator

$$\hat{T}(a_1)\hat{O}(a_1)\hat{T}^\dagger(a_1) = \hat{O}(a_2) \tag{2.2}$$

and the operators

$$\hat{B}_+ = \hat{A}^\dagger(a_1)\hat{T}(a_1) \qquad \text{and} \qquad \hat{B}_- = \hat{B}_+^\dagger = \hat{T}^\dagger(a_1)\hat{A}(a_1) \tag{2.3}$$

the Hamiltonians of equation (1.1) take the forms

$$\hat{H}_1 = \hbar\Omega\hat{B}_+\hat{B}_- \qquad \text{and} \qquad \hat{H}_2 = \hbar\Omega\hat{T}\hat{B}_-\hat{B}_+\hat{T}^\dagger. \tag{2.4}$$

With equation (2.1) one can also easily prove the commutation relation [4]

$$[\hat{B}_-, \hat{B}_+] = \hat{T}^\dagger(a_1)R(a_1)\hat{T}(a_1) \equiv R(a_0) \tag{2.5}$$

where we used the identity

$$R(a_n) = \hat{T}(a_1)R(a_{n-1})\hat{T}^\dagger(a_1) \tag{2.6}$$

valid for any n. Equation (2.5) suggests that $\hat{B}_-$ and $\hat{B}_+$ are the appropriate creation and annihilation operators for the spectra of the shape-invariant potentials provided that their non-commutativity with $R(a_1)$ is taken into account. Indeed using the relations

$$R(a_n)\hat{B}_+ = \hat{B}_+R(a_{n-1}) \qquad \text{and} \qquad R(a_n)\hat{B}_- = \hat{B}_+R(a_{n+1}) \tag{2.7}$$

which readily follow from equations (2.3) and (2.6), one can write the additional commutation relations

$$\begin{cases}[\hat{B}_+, R(a_0)] = \{R(a_1) - R(a_0)\}\hat{B}_+ \\ [\hat{B}_+, \{R(a_1) - R(a_0)\}] = \{R(a_2) - R(a_0)\}\hat{B}_+^2\end{cases} \tag{2.8}$$

and the Hermitian conjugate relations. In general there is an infinite number of such commutation relations, hence the appropriate Lie algebra is infinite dimensional. The ground state of the Hamiltonian $\hat{H}_1$ satisfies the condition

$$\hat{A}|\Psi_0\rangle = 0 = \hat{B}_-|\Psi_0\rangle. \tag{2.9}$$

The normalized nth excited state of $\hat{H}_1$ is

$$|\Psi_n\rangle = \frac{1}{\sqrt{R(a_1) + R(a_2) + \cdots + R(a_n)}}\hat{B}_+ \cdots \frac{1}{\sqrt{R(a_1) + R(a_2)}}\hat{B}_+ \frac{1}{\sqrt{R(a_1)}}\hat{B}_+|\Psi_0\rangle \tag{2.10}$$

with the eigenvalue $E_n = \hbar\Omega e_n$, where

$$e_n = \sum_{k=1}^{n} R(a_k). \tag{2.11}$$

The action of the $\hat{B}_\pm$ operators on that state given in equation (2.10) is

$$\hat{B}_+|\Psi_n\rangle = \sqrt{e_{n+1}}|\Psi_{n+1}\rangle \qquad \text{and} \qquad \hat{B}_-|\Psi_n\rangle = \sqrt{e_{n-1} + R(a_0)}|\Psi_{n-1}\rangle. \tag{2.12}$$

Finally, using equations (2.12) it is easy to show that

$$\hat{H}_1|\Psi_n\rangle \equiv \hbar\Omega\left(\hat{B}_+\hat{B}_-\right)|\Psi_n\rangle = \hbar\Omega e_n|\Psi_n\rangle \qquad \text{and} \qquad \hat{B}_-\hat{B}_+|\Psi_n\rangle = \{e_n + R(a_0)\}|\Psi_n\rangle. \tag{2.13}$$

3. Quantum deformation of shape-invariant systems

3.1. q-deformed creation and annihilation operators

To obtain a quantum-deformed version of the shape-invariant systems Hamiltonian we introduce the q-deformed forms for the creation and annihilation operators $\hat{B}_-^{(q)}$ and $\hat{B}_+^{(q)}$ in terms of the usual $\hat{B}_\pm$ operators as

$$\hat{B}_-^{(q)} \equiv \sqrt{\frac{[\hat{B}_-\hat{B}_+]_q}{\hat{B}_-\hat{B}_+}}\hat{B}_- = \hat{B}_-\sqrt{\frac{[\hat{B}_+\hat{B}_-]_q}{\hat{B}_+\hat{B}_-}} \tag{3.1}$$

and

$$\hat{B}_+^{(q)} \equiv (\hat{B}_-^{(q)})^\dagger = \hat{B}_+\sqrt{\frac{[\hat{B}_-\hat{B}_+]_q}{\hat{B}_-\hat{B}_+}} = \sqrt{\frac{[\hat{B}_+\hat{B}_-]_q}{\hat{B}_+\hat{B}_-}}\hat{B}_+ \tag{3.2}$$

where we used the definition

$$[x]_q \equiv \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{3.3}$$

For the limit $q \to 1$, it is easy to show that

$$\lim_{q\to 1}[x]_q = x \tag{3.4}$$

and the postulated relations (3.1) and (3.2) tend to the usual generalized ladder operators. For any analytical function $f(x)$ it is easy to show that

$$\hat{B}_\pm f(\hat{B}_\mp\hat{B}_\pm) = f(\hat{B}_\pm\hat{B}_\mp)B_\pm. \tag{3.5}$$

In particular we have

$$\hat{B}_\pm\left[\frac{q^{\pm\hat{B}_\mp\hat{B}_\pm}}{\hat{B}_\mp\hat{B}_\pm}\right] = \left[\frac{q^{\pm\hat{B}_\pm\hat{B}_\mp}}{\hat{B}_\pm\hat{B}_\mp}\right]\hat{B}_\pm. \tag{3.6}$$

3.2. *q-deformed Hamiltonian*

With our definition of q-deformed ladder operators we can write the q-deformed form of the Hamiltonian $\hat{H}_1$ as

$$\hat{H}_1^{(q)} = \hbar\Omega\big(\hat{B}_+^{(q)}\hat{B}_-^{(q)}\big) = \hbar\Omega\left(\sqrt{\frac{[\hat{B}_+\hat{B}_-]_q}{\hat{B}_+\hat{B}_-}}\hat{B}_+\hat{B}_-\sqrt{\frac{[\hat{B}_+\hat{B}_-]_q}{\hat{B}_+\hat{B}_-}}\right) = \hbar\Omega[\hat{B}_+\hat{B}_-]_q \tag{3.7}$$

and also

$$\hat{B}_-^{(q)}\hat{B}_+^{(q)} = \sqrt{\frac{[\hat{B}_-\hat{B}_+]_q}{\hat{B}_-\hat{B}_+}}\hat{B}_-\hat{B}_+\sqrt{\frac{[\hat{B}_-\hat{B}_+]_q}{\hat{B}_-\hat{B}_+}} = [\hat{B}_-\hat{B}_+]_q. \tag{3.8}$$

With these results we conclude that the commutator of the q-deformed $\hat{B}$ operators is given by

$$\big[\hat{B}_-^{(q)}, \hat{B}_+^{(q)}\big] = [\hat{B}_-\hat{B}_+]_q - [\hat{B}_+\hat{B}_-]_q. \tag{3.9}$$

Another conclusion is that the q-deformed version of the Hamiltonian and its undeformed version commute with each other, $\big[\hat{H}_1^{(q)}, \hat{H}_1\big] = 0$, therefore they have the common set of eigenstates give by equation (2.10). Then taking into account equations (2.12), (2.13), (3.1) and (3.2) we can show that

$$\hat{B}_+^{(q)}|\Psi_n\rangle = \sqrt{[e_{n+1}]_q}|\Psi_{n+1}\rangle \qquad \text{and} \qquad \hat{B}_-^{(q)}|\Psi_n\rangle = \sqrt{[e_{n-1} + R(a_0)]_q}|\Psi_{n-1}\rangle. \tag{3.10}$$

To get the eigenvalues of $\hat{H}_1^{(q)}$ we can use these results and equations (3.7) and (3.8) to obtain

$$\begin{aligned} &\hat{H}_1^{(q)}|\Psi_n\rangle = \hbar\Omega\hat{B}_+^{(q)}\hat{B}_-^{(q)}|\Psi_n\rangle = \hbar\Omega[e_n]_q|\Psi_n\rangle \qquad \text{and} \\ &\hat{B}_-^{(q)}\hat{B}_+^{(q)}|\Psi_n\rangle = [e_n + R(a_0)]_q|\Psi_n\rangle. \end{aligned} \tag{3.11}$$

3.3. *Generalized quantum deformed models for shape-invariant potentials*

With the definitions presented in the previous section it is possible to define new q-deformed ladder operators and their q-commutations relations as we illustrate in this section.

3.3.1. Standard model. Using equations (3.7), (3.8) and the commutation relation (2.5) we can evaluate the product

$$\begin{aligned} \hat{B}_-^{(q)}\hat{B}_+^{(q)} - q^{R(a_0)}\hat{B}_+^{(q)}\hat{B}_-^{(q)} &= [\hat{B}_-\hat{B}_+]_q - q^{R(a_0)}[\hat{B}_+\hat{B}_-]_q \\ &= \frac{1}{q-q^{-1}}\{q^{\hat{B}_-\hat{B}_+} - q^{-\hat{B}_-\hat{B}_+} - q^{(\hat{B}_-\hat{B}_+ - \hat{B}_+\hat{B}_-)}[q^{\hat{B}_+\hat{B}_-} - q^{-\hat{B}_+\hat{B}_-}]\} \\ &= \frac{q^{-\hat{B}_+\hat{B}_-}}{q-q^{-1}}\{q^{(\hat{B}_-\hat{B}_+ - \hat{B}_+\hat{B}_-)} - q^{(-\hat{B}_-\hat{B}_+ + \hat{B}_+\hat{B}_-)}\} \end{aligned} \tag{3.12}$$

which gives

$$\hat{B}_-^{(q)}\hat{B}_+^{(q)} - q^{R(a_0)}\hat{B}_+^{(q)}\hat{B}_-^{(q)} = [R(a_0)]_q q^{-\hat{B}_+\hat{B}_-}. \tag{3.13}$$

In a similar way we can show that

$$\hat{B}_-^{(q)}\hat{B}_+^{(q)} - q^{-R(a_0)}\hat{B}_+^{(q)}\hat{B}_-^{(q)} = [R(a_0)]_q q^{\hat{B}_+\hat{B}_-} \tag{3.14}$$

an expected result considering the invariance under the substitution of $q \to q^{-1}$ of the q-number definition (3.3). Equations (3.13) and (3.14) represent q-commutators for any shape-invariant potential. In the particular case of the harmonic oscillator potential we have $R(a_0) = R(a_1) = \cdots = \text{cte.} = 1$ after a suitable normalization and obtain

$$\hat{B}_-^{(q)} \longrightarrow \hat{a}_q = \sqrt{\frac{[\hat{N}-1]_q}{\hat{N}-1}}\hat{a} = \hat{a}\sqrt{\frac{[\hat{N}]_q}{\hat{N}}} \qquad \hat{B}_+^{(q)} \longrightarrow \hat{a}_q^\dagger = \hat{a}^\dagger\sqrt{\frac{[\hat{N}-1]_q}{\hat{N}-1}} = \sqrt{\frac{[\hat{N}]_q}{\hat{N}}}\hat{a}^\dagger. \tag{3.15}$$

Therefore, equations (3.13) and (3.14) reduce to the form

$$\hat{a}_q\hat{a}_q^\dagger - q^{\pm 1}a_q^\dagger\hat{a}_q = q^{\mp\hat{N}} \tag{3.16}$$

where $\hat{N} = \hat{a}^\dagger\hat{a} \neq \hat{a}_q^\dagger\hat{a}_q$. The operators $\hat{a}_q$ and $\hat{a}_q^\dagger$ and its q-deformed commutation relation (3.16) are the basic assumptions usually postulated in the study of the standard q-deformed harmonic oscillator models [10, 17, 18]. The relation (3.16) also is termed q-deformed *physics* boson canonical commutation relation [19] and was introduced in order to provide a realization of quantum groups [17, 18] which arise naturally in the solution of certain lattice models [8].

3.3.2. Generalized Q-deformed models. A second way to construct a q-deformed model for a shape-invariant potential can be obtained if we define the new operators

$$\begin{cases} \hat{C}_-^{(q)} \equiv \frac{1}{\sqrt{q}}\hat{B}_-^{(q)}q^{(\hat{B}_+\hat{B}_-)/2} = \frac{1}{\sqrt{q}}q^{(\hat{B}_-\hat{B}_+)/2}\hat{B}_-^{(q)} \\ \hat{C}_+^{(q)} = \hat{C}_-^{(q)\dagger} = \frac{1}{\sqrt{q}}q^{(\hat{B}_+\hat{B}_-)/2}\hat{B}_+^{(q)} = \frac{1}{\sqrt{q}}\hat{B}_+^{(q)}q^{(\hat{B}_-\hat{B}_+)/2}. \end{cases} \tag{3.17}$$

Using the results of equations (3.7), (3.8), the commutation relation (2.5) and the commutation between any function of the remainders $R(a_n)$, and the couple of operators $\hat{B}_\pm\hat{B}_\mp$ we can evaluate the products

$$\begin{aligned} \hat{C}_-^{(q)}\hat{C}_+^{(q)} &= \frac{1}{\sqrt{q}}q^{(\hat{B}_-\hat{B}_+)/2}\hat{B}_-^{(q)}\hat{B}_+^{(q)}q^{(\hat{B}_-\hat{B}_+)/2}\frac{1}{\sqrt{q}} \\ &= q^{\hat{B}_-\hat{B}_+}[\hat{B}_-\hat{B}_+]_q/q \\ &= \frac{q^{2\hat{B}_-\hat{B}_+}-1}{q^2-1} \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} \hat{C}_+^{(q)}\hat{C}_-^{(q)} &= \frac{1}{\sqrt{q}}q^{(\hat{B}_+\hat{B}_-)/2}\hat{B}_+^{(q)}\hat{B}_-^{(q)}q^{(\hat{B}_+\hat{B}_-)/2}\frac{1}{\sqrt{q}} \\ &= q^{\hat{B}_+\hat{B}_-}[\hat{B}_+\hat{B}_-]_q \\ &= \frac{q^{2\hat{B}_+\hat{B}_-}-1}{q^2-1}. \end{aligned} \tag{3.19}$$

With the results found in equations (3.18) and (3.19), and the commutation relation (2.5) it is possible to establish the q-deformed commutation relation

$$\hat{C}_-^{(q)}\hat{C}_+^{(q)} - q^{2R(a_0)}\hat{C}_+^{(q)}\hat{C}_-^{(q)} = q^{R(a_0)}[R(a_0)]_q/q. \tag{3.20}$$

The definition of q-numbers given by equation (3.3) is not the only possible one. There is an alternative definition called the Q-numbers. Indeed if we change $q^2 \longrightarrow Q$ and use Q-operators generalization of the Q-numbers definition

$$[x]_q = \frac{Q^x - 1}{Q - 1} \tag{3.21}$$

it is possible to rewrite definitions (3.17) as

$$\hat{C}_{\pm}^{(q)} = \hat{B}_{\pm}^{(Q)} = \sqrt{\frac{[(\hat{B}_{\pm}\hat{B}_{\mp}]_Q}{\hat{B}_{\pm}\hat{B}_{\mp}}}\hat{B}_{\pm} = \hat{B}_{\pm}\sqrt{\frac{[\hat{B}_{\mp}\hat{B}_{\pm}]_Q}{\hat{B}_{\mp}\hat{B}_{\pm}}} \tag{3.22}$$

and show that the q-deformed commutation relation (3.20) can be written in its Q-deformed version as

$$\hat{B}_{-}^{(Q)}\hat{B}_{+}^{(Q)} - Q^{R(a_0)}\hat{B}_{+}^{(Q)}\hat{B}_{-}^{(Q)} = [R(a_0)]_Q. \tag{3.23}$$

Again, for a harmonic oscillator potential system we have that $R(a_0) = R(a_1) = \cdots = 1$ and

$$\begin{aligned} \hat{C}_{-}^{(q)} &\longrightarrow \hat{b}_Q = \hat{a}\sqrt{\frac{[\hat{N}]_Q}{\hat{N}}} = \sqrt{\frac{[\hat{N}-1]_Q}{\hat{N}-1}}\hat{a} \\ \hat{C}_{+}^{(q)} &\longrightarrow \hat{b}_Q^{\dagger} = \hat{a}^{\dagger}\sqrt{\frac{[\hat{N}-1]_Q}{\hat{N}-1}} = \sqrt{\frac{[\hat{N}]_Q}{\hat{N}}}\hat{a}^{\dagger}. \end{aligned} \tag{3.24}$$

In this case equation (3.20) reduces to the form

$$\hat{b}_Q\hat{b}_Q^{\dagger} - Qb_Q^{\dagger}\hat{b}_Q = 1 \tag{3.25}$$

that together with the definition of $\hat{b}_Q$ operators corresponds to a different version of the deformed harmonic oscillator model, first introduced by Arik and Coon [20] and later also considered by Kuryshkin [21]. Thus, we can consider the operators $\hat{B}_{\pm}^{(Q)}$ and equation (3.23) as the generalized version for all shape-invariant systems of the Q-deformed basic relations first postulated by Arik and Coon for the Q-deformed harmonic oscillator model.

3.3.3. Another generalized q-deformed model. Another q-deformed model can be obtained if we define the new operators

$$\begin{cases} \hat{D}_{-}^{(q)} = q^{-R(a_0)/2}\hat{B}_{-}^{(q)}q^{\hat{B}_{+}\hat{B}_{-}/2} = q^{-R(a_0)/2}q^{\hat{B}_{-}\hat{B}_{+}/2}\hat{B}_{-}^{(q)} \\ \hat{D}_{+}^{(q)} = \hat{D}_{-}^{(q)\dagger} = q^{\hat{B}_{+}\hat{B}_{-}/2}\hat{B}_{+}^{(q)}q^{-R(a_0)/2} = \hat{B}_{+}^{(q)}q^{\hat{B}_{-}\hat{B}_{+}/2}q^{-R(a_0)/2}. \end{cases} \tag{3.26}$$

Using the results of equations (3.7) and (3.8), the commutation relation (2.5) and the commutation between any function of the remainders $R(a_n)$, and the couple of operators $\hat{B}_{\pm}\hat{B}_{\mp}$ we can write the product

$$\begin{aligned} \hat{D}_{-}^{(q)}\hat{D}_{+}^{(q)} &= q^{-R(a_0)/2}q^{\hat{B}_{-}\hat{B}_{+}/2}\hat{B}_{-}^{(q)}\hat{B}_{+}^{(q)}q^{\hat{B}_{-}\hat{B}_{+}/2}q^{-R(a_0)/2} \\ &= q^{-R(a_0)}q^{\hat{B}_{-}\hat{B}_{+}}[\hat{B}_{-}\hat{B}_{+}]_q \\ &= \frac{q^{(\hat{B}_{-}\hat{B}_{+}+\hat{B}_{+}\hat{B}_{-})} - q^{-R(a_0)}}{q-q^{-1}} \end{aligned} \tag{3.27}$$

and, with the help of equations (2.7), the product

$$\begin{aligned} \hat{D}_{+}^{(q)}\hat{D}_{-}^{(q)} &= q^{\hat{B}_{+}\hat{B}_{-}/2}\hat{B}_{+}^{(q)}q^{-R(a_0)}\hat{B}_{-}^{(q)}q^{\hat{B}_{+}\hat{B}_{-}/2} \\ &= q^{-R(a_1)}q^{\hat{B}_{+}\hat{B}_{-}}[\hat{B}_{+}\hat{B}_{-}]_q \\ &= q^{-[R(a_0)+R(a_1)]}\left(\frac{q^{(\hat{B}_{+}\hat{B}_{-}+\hat{B}_{-}\hat{B}_{+})} - q^{R(a_0)}}{q-q^{-1}}\right). \end{aligned} \tag{3.28}$$

Using the results shown in equations (3.27) and (3.28) it is possible to establish the following q-deformed commutation relation:

$$\hat{D}_{-}^{(q)}\hat{D}_{+}^{(q)} - q^{[R(a_0)+R(a_1)]}\hat{D}_{+}^{(q)}\hat{D}_{-}^{(q)} = [R(a_0)]_q. \tag{3.29}$$

In the limiting case of a harmonic oscillator potential system, when $R(a_0) = R(a_1) = \cdots = 1$ one has

$$\hat{D}_-^{(q)} \longrightarrow \hat{b}_q = q^{-1/2}\hat{a}_q q^{\hat{N}/2} = q^{\hat{N}/2}\hat{a}_q \qquad \hat{D}_+^{(q)} \longrightarrow \hat{b}_q^\dagger = q^{-1/2}q^{\hat{N}/2}\hat{a}_q^\dagger = \hat{a}_q^\dagger q^{\hat{N}/2} \tag{3.30}$$

which gives for equation (3.29) the following form:

$$\hat{b}_q\hat{b}_q^\dagger - q^2 b_q^\dagger \hat{b}_q = 1. \tag{3.31}$$

Hence, there are two possible and different shape-invariant generalizations for the Arik and Coon quantum-deformed model. The first by using the operators $\hat{B}_\pm^{(Q)}$ and equation (3.23), and the second with the operators $\hat{D}_\pm^{(q)}$ and equation (3.29). These two generalizations for shape-invariant systems are equivalent when we apply to the harmonic oscillator potential system, giving the standard Arik and Coon model.

An important aspect to observe at this point is that all quantum-deformed models generalized from the primary shape-invariant potentials and presented in these previous sections do not preserve the shape invariance after the quantum deformation. In other words, the quantum deformation breaks the shape invariance of the final q-deformed system. Obviously this fact is a result of the basic assumptions used to build the quantum-deformed models.

4. Shape-invariant quantum-deformed systems

The purpose of this section is to build an alternative generalized quantum-deformed model which, unlike the previous ones, after the quantum deformation, preserves the shape-invariance condition shown by the primary system.

4.1. New q-deformed ladder operators

To find a q-deformed system formulation which preserves the shape-invariant condition we introduce new operators defined by

$$\begin{cases} \hat{S}_-^{(q)} = \mathcal{F}\hat{B}_-^{(q)} q^{\hat{B}_+\hat{B}_-/2} = \mathcal{F} q^{\hat{B}_-\hat{B}_+/2}\hat{B}_-^{(q)} \\ \hat{S}_+^{(q)} = \hat{S}_-^{(q)\dagger} = q^{\hat{B}_+\hat{B}_-/2}\hat{B}_+^{(q)}\mathcal{F} = \hat{B}_+^{(q)} q^{\hat{B}_-\hat{B}_+/2}\mathcal{F} \end{cases} \tag{4.1}$$

where the $\hat{B}_\pm^{(q)}$ operators were introduced by equations (3.2), (3.1) and $\mathcal{F}$ is a compact notation for a real functional of the potential parameters $a_0, a_1, a_2, \ldots$. Note that for the Hermitian conjugation condition written above to be satisfied, q must be assumed as a real parameter. We specify the conditions on $\mathcal{F}$ below. Considering that $[\mathcal{F}, \hat{B}_\pm\hat{B}_\mp] = 0$ and using the definitions in equation (4.1), the commutation relations (2.5) and equations (3.7), (3.8) and (2.7), it is possible to evaluate the products

$$\begin{aligned} \hat{S}_-^{(q)}\hat{S}_+^{(q)} &= \mathcal{F} q^{\hat{B}_-\hat{B}_+/2}\hat{B}_-^{(q)}\hat{B}_+^{(q)} q^{\hat{B}_-\hat{B}_+/2}\mathcal{F} \\ &= \mathcal{F}^2 q^{\hat{B}_-\hat{B}_+}[\hat{B}_-\hat{B}_+]_q \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \hat{S}_+^{(q)}\hat{S}_-^{(q)} &= q^{\hat{B}_+\hat{B}_-/2}\hat{B}_+^{(q)}\mathcal{F}^2\hat{B}_-^{(q)} q^{\hat{B}_+\hat{B}_-/2} \\ &= \hat{T}(a_1)\mathcal{F}^2\hat{T}^\dagger(a_1) q^{\hat{B}_+\hat{B}_-}[\hat{B}_+\hat{B}_-]_q. \end{aligned} \tag{4.3}$$

Now, with the results of equations (4.2), (4.3) and the commutation relation (2.5) we can write the commutator of the $\hat{S}$-operators

$$\left[\hat{S}_-^{(q)}, \hat{S}_+^{(q)}\right] = \mathcal{F}^2 q^{R(a_0)} q^{\hat{B}_+\hat{B}_-}[\hat{B}_-\hat{B}_+]_q - \hat{T}(a_1)\mathcal{F}^2\hat{T}^\dagger(a_1) q^{-R(a_0)} q^{\hat{B}_-\hat{B}_+}[\hat{B}_+\hat{B}_-]_q. \tag{4.4}$$

At this point, we assume that the functional operator $\mathcal{F}$, till now considered arbitrary, satisfies the constraint

$$\hat{T}(a_1)\mathcal{F}^2 T^\dagger(a_1) = q^{2R(a_0)}\mathcal{F}^2. \tag{4.5}$$

Thus, taking into account this condition and the operator relation

$$q^{\hat{B}_\pm \hat{B}_\mp}[\hat{B}_\mp \hat{B}_\pm]_q = \frac{q^{(\hat{B}_+\hat{B}_- + \hat{B}_-\hat{B}_+)} - q^{\mp R(a_0)}}{q - q^{-1}} \tag{4.6}$$

it follows that the commutator can be written as

$$\left[\hat{S}_-^{(q)}, \hat{S}_+^{(q)}\right] = \mathcal{G}_0 \qquad \text{with} \qquad \mathcal{G}_0 \equiv \mathcal{F}^2 q^{R(a_0)}\,[R(a_0)]_q. \tag{4.7}$$

Comparing equations (2.5) and (4.7) we conclude that the later can be associated with a shape-invariance condition as the former and that $\hat{S}_-^{(q)}$ and $\hat{S}_+^{(q)}$ are the appropriate creation, and annihilation operators for the spectra of the q-deformed shape-invariant systems whose Hamiltonian and its eigenstates and eigenvalues will be determined in the following section.

4.2. Hamiltonian, eigenstates and eigenvalues

Using the new ladder operators introduced in the previous section we can define a new Hamiltonian as

$$\hat{\mathcal{H}}^{(q)} = \hbar\Omega \hat{S}_+^{(q)} \hat{S}_-^{(q)}. \tag{4.8}$$

With this definition, relations (2.7) and equation (4.7) we can write the additional commutation relations

$$\left[\hat{\mathcal{H}}^{(q)}, \left(\hat{S}_+^{(q)}\right)^n\right] = \hbar\Omega\{\mathcal{G}_1 + \mathcal{G}_2 + \cdots + \mathcal{G}_n\}\left(\hat{S}_+^{(q)}\right)^n \tag{4.9}$$

and

$$\left[\hat{\mathcal{H}}^{(q)}, \left(\hat{S}_-^{(q)}\right)^n\right] = -\hbar\Omega\left(\hat{S}_-^{(q)}\right)^n\{\mathcal{G}_1 + \mathcal{G}_2 + \cdots + \mathcal{G}_n\} \tag{4.10}$$

where we defined

$$\mathcal{G}_n = T(a_1)\mathcal{G}_{n-1}T^\dagger(a_1) \tag{4.11}$$

with $\mathcal{G}_0$ given by equation (4.7). Using equations (2.9), (3.1) and (4.1) we can also show that $\hat{S}_-^{(q)}|\Psi_0\rangle = 0$. From this result and commutator (4.9) it follows:

$$\hat{\mathcal{H}}^{(q)}\left\{\left(\hat{S}_+^{(q)}\right)^n|\Psi_0\rangle\right\} = \hbar\Omega\left\{\mathcal{G}_1 + \mathcal{G}_2 + \cdots + \mathcal{G}_n\right\}\left\{\left(\hat{S}_+^{(q)}\right)^n|\Psi_0\rangle\right\} \tag{4.12}$$

i.e., $\left(\hat{S}_+^{(q)}\right)^n|\Psi_0\rangle$ is an eigenstate of the Hamiltonian $\hat{\mathcal{H}}^{(q)}$ with the eigenvalue

$$\mathcal{E}_n = \hbar\Omega\sum_{k=1}^{n}\mathcal{G}_k. \tag{4.13}$$

Indeed this conclusion could be obtained with another approach. With the operator definition in equation (4.1) and the functional condition in equation (4.5), the Hamiltonian $\hat{\mathcal{H}}^{(q)}$ can be written in terms of the primary shape-invariant operators $\hat{B}_\pm$ as

$$\hat{\mathcal{H}}^{(q)} = \hbar\Omega q^{2R(a_0)}\mathcal{F}^2 q^{\hat{B}_+\hat{B}_-}[\hat{B}_+\hat{B}_-]_q. \tag{4.14}$$

Obviously we have that $[\hat{\mathcal{H}}^{(q)}, \hat{H}_1] = 0$, hence these two Hamiltonians have a common set of eigenstates give by equation (2.10), in other words $|\Psi_n\rangle \propto \left(\hat{S}_+^{(q)}\right)^n|\Psi_0\rangle$. Now, to get the eigenvalues of $\hat{\mathcal{H}}^{(q)}$ it is enough to use this expression, equations (2.13) and (3.11) to obtain

$$\hat{\mathcal{H}}^{(q)}|\Psi_n\rangle = \hbar\Omega q^{2R(a_0)}\mathcal{F}^2 q^{e_n}[e_n]_q|\Psi_n\rangle \tag{4.15}$$

where e_n is given by equation (2.11). Indeed, by using the generalization of condition in equation (4.5)

$$\{\hat{T}(a_1)\}^k \mathcal{F} \{\hat{T}^\dagger(a_1)\}^k = \prod_{j=0}^{k-1} q^{R(a_j)} \mathcal{F} \tag{4.16}$$

it is straightforward to show that

$$\mathcal{E}_n = \hbar\Omega \sum_{k=1}^{n} \mathcal{G}_k = \hbar\Omega \sum_{k=1}^{n} \{\hat{T}(a_1)\}^k \mathcal{F} \{\hat{T}^\dagger(a_1)\}^k q^{R(a_k)} [R(a_k)]_q = \hbar\Omega q^{2R(a_0)} \mathcal{F}^2 q^{e_n} [e_n]_q. \tag{4.17}$$

On the other hand, with equations (2.7), (2.13), (3.10), (4.1) and condition (4.5) it is possible to show that

$$\hat{S}_+^{(q)} |\Psi_n\rangle = q^{R(a_0)} \mathcal{F} q^{e_{n+1}/2} \sqrt{[e_{n+1}]_q} |\Psi_{n+1}\rangle \tag{4.18}$$

and

$$\hat{S}_-^{(q)} |\Psi_n\rangle = q^{R(a_0)/2} \mathcal{F} q^{e_{n-1}/2} \sqrt{[e_{n-1} + R(a_0)]_q} |\Psi_{n-1}\rangle \tag{4.19}$$

showing clearly that $\hat{S}_+^{(q)}$ and $\hat{S}_-^{(q)}$ represent appropriate creation and annihilation operators.

In addition to the commutation relations in equations (4.9) and (4.10) we can establish the commutation relations

$$\left[\hat{S}_+^{(q)}, \mathcal{G}_j\right] = \{\mathcal{G}_{j+1} - \mathcal{G}_j\} \hat{S}_+^{(q)} \tag{4.20}$$

$$\left[\hat{S}_+^{(q)}, \{\mathcal{G}_{j+1} - \mathcal{G}_j\}\right] = \{\mathcal{G}_{j+2} - \mathcal{G}_j\} \left(\hat{S}_+^{(q)}\right)^2 \tag{4.21}$$

and so on. In general, there is an infinite number of these commutation relations, which with their complex conjugates together with equation (4.7) form an infinite-dimensional Lie algebra, realized here in an unitary representation.

4.3. An ansatz for the functional $\mathcal{F}$

Until this point we did not specify the functional $\mathcal{F}$. In this section we will give an ansatz for $\mathcal{F}$. We will begin by noting that the remainder $R(a_j)$ is given by the expression

$$R(a_j) = C[f(a_j) - f(a_{j+1})] \tag{4.22}$$

for a number of shape-invariant potentials. In equation (4.22) both the constant C and the function $f(a_j)$ are determined by the particular potential in consideration.

For the Morse potential, $V(x) = V_0(\mathrm{e}^{-2\lambda x} - 2b\,\mathrm{e}^{-\lambda x})$, the superpotential is [4]

$$W(x; a_n) = \sqrt{V_0}(a_n - \mathrm{e}^{-\lambda x}). \tag{4.23}$$

$R(a_j)$ is given by

$$R(a_j) = \left[a_j^2 - a_{j+1}^2\right] \tag{4.24}$$

with

$$a_j = b - \frac{\lambda\hbar}{\sqrt{2mV_0}} \left(j - \frac{1}{2}\right) \tag{4.25}$$

where we identified $\hbar\Omega \equiv V_0$. Hence we get $f(a_j) = a_j^2$.

For the Scarf potential, $V(x) = -V_0/\cosh^2 \lambda x$, the superpotential is [4]

$$W(x; a_n) = \sqrt{V_0} a_n \tanh \lambda x. \tag{4.26}$$

$R(a_j)$ is given by

$$R(a_j) = \left[a_j^2 - a_{j+1}^2\right] \tag{4.27}$$

with

$$a_j = \frac{1}{2}\left(\sqrt{\frac{8mV_0}{\hbar^2\lambda^2} + 1} - 2j + 1\right) \tag{4.28}$$

if we identify $\Omega = \lambda\sqrt{V_0/(2m)}$. Hence one again obtains $f(a_j) = a_j^2$.

For the Coulomb potential, $V_L(r) = -\frac{Ze^2}{r} + \frac{\hbar^2\lambda^2}{2m}L(L+1)$, we have $a_L = L$. The superpotential is [22]

$$W(r; L) = \sqrt{\frac{m(Z\,\mathrm{e}^2)^2}{2\hbar^2}}\left(\frac{1}{(L+1)} - \frac{\hbar^2}{mZ\,\mathrm{e}^2}\frac{L+1}{r}\right) \tag{4.29}$$

with

$$R(L) = \left[\frac{1}{(L+1)^2} - \frac{1}{(L+2)^2}\right] \tag{4.30}$$

if we identify $\hbar\Omega = m(Z\,\mathrm{e}^2)^2/(2\hbar^2)$. In this case $f(L) = \frac{1}{(L+1)^2}$ or alternatively $f(a_j) = \frac{1}{(a_j+1)^2}$.

We can give an explicit ansatz for $\mathcal{F}$ applicable to those cases where equation (4.22) is satisfied. Noting the identity

$$\hat{T}(a_1)q^{-2Cf(a_0)}\hat{T}^\dagger(a_1) = q^{-2Cf(a_1)} = q^{2R(a_0)}q^{-2Cf(a_0)} \tag{4.31}$$

we conclude that in those cases the functional $\mathcal{F}$ depends only on a_0

$$\mathcal{F} = q^{-Cf(a_0)}. \tag{4.32}$$

For the shape-invariant potentials that satisfy equation (4.22) we can then define

$$\mathcal{F}_0 \equiv q^{-Cf(a_0)} \tag{4.33}$$

and

$$\mathcal{F}_n = T(a_1)\mathcal{F}_{n-1}T^\dagger(a_1). \tag{4.34}$$

At the moment a more general expression for $\mathcal{F}$ is not readily available.

5. Conclusions

In this paper we introduced a quantum-deformed theory applicable to all shape-invariant systems. To achieve this we introduced the appropriate q-deformed ladder operators. We also constructed an alternative q-deformed model that preserves the shape-invariance property presented by the primary system. Our results are applicable to those shape-invariant potentials where the potential parameters are related by a translation.

Shape-invariance represents the exact solvability of a system. We have previously given a method of obtaining a new exactly solvable system starting with a known shape-invariant system by coupling it to a two-level system. The resulting models generalize the Jaynes–Cummings model by substituting a shape-invariant system instead of the harmonic oscillator [23, 24]. The results presented in this paper represent generalization in a different direction, that is, deformation not of the harmonic oscillator, but of certain shape-invariant systems.

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References

[1] Witten E 1981 *Nucl. Phys.* B **188** 513
[2] Cooper F, Khare A and Sukhatme U 1995 *Phys. Rep.* **251** 267 (*Preprint* hep-th/9405029)
[3] Gendenshtein L E 1983 *JETP Lett.* **38** 356
Gendenshtein L E 1983 *Pisma Zh. Eksp. Teor. Fiz.* **38** 299
[4] Balantekin A B 1998 *Phys. Rev.* A **57** 4188 (*Preprint* quant-ph/9712018)
[5] Drinfeld V G 1988 *J. Sov. Math.* **41** 898
Drinfeld V G 1986 *Zap. Nauchn. Semin.* **155** 18
[6] Jimbo M 1986 *Lett. Math. Phys.* **11** 247
[7] Jimbo M 1989 *Adv. Ser. Math. Phys.* **10** 1
[8] Batchelor M T, Mezincescu L, Nepomechie R I and Rittenberg V 1990 *J. Phys. A: Math. Gen.* **23** L141
Grosse H, Pallua S, Prester P and Raschhofer E 1994 *J. Phys. A: Math. Gen.* **27** 4761
Ubriaco M R 1997 *Phys. Rev.* E **55** 291 (*Preprint* hep-th/9610124)
[9] Barbier R, Meyer J and Kibler M 1994 *J. Phys. G: Nucl. Part. Phys.* **20** L13 (*Preprint* hep-th/9407054)
Angelova Maia, Dobrev V K and Frank A 2001 *J. Phys. A: Math. Gen.* **34** L503
[10] Bonatsos D and Daskaloyannis C 1999 *Prog. Part. Nucl. Phys.* **43** 537 (*Preprint* nucl-th/9909003)
[11] Chaichian M, Ellinas D and Kulish P 1990 *Phys. Rev. Lett.* **65** 980
Buzek V 1992 *J. Mod. Opt.* **39** 949
Kuang L M 1994 *J. Mod. Opt.* **41** 517
Marzlin K P 1997 *Int. J. Thermophys.* **36** 841
Zhe C 1993 *Phys. Rev.* A **47** 5017
[12] Alvarez-Gaume L, Gomez C and Sierra G 1989 *Phys. Lett.* B **220** 142
Alvarez-Gaume L, Gomez C and Sierra G 1990 *Nucl. Phys.* B **330** 347
[13] Chuan C 1991 *J. Phys. A: Math. Gen.* **24** L1165
[14] Balantekin A B, Candido Ribeiro M A and Aleixo A N 1999 *J. Phys. A: Math. Gen.* **32** 2785 (*Preprint* quant-ph/9811061)
[15] Aleixo A N, Balantekin A B and Candido Ribeiro M A 2002 *J. Phys. A: Math. Gen.* **35** 9063 (*Preprint* math-ph/0209033)
[16] Khare A and Sukhatme U P 1993 *J. Phys. A: Math. Gen.* **26** L901 (*Preprint* hep-th/9212147)
Barclay D T, Dutt R, Gangopadhyaya A, Khare A, Pagnamenta A and Sukhatme U 1993 *Phys. Rev.* A **48** 2786 (*Preprint* hep-ph/9304313)
[17] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
[18] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
[19] Solomon A I 1994 *Phys. Lett.* A **196** 29
[20] Iwata G 1951 *Prog. Theor. Phys.* **6** 524
Arik M and Coon D D 1976 *J. Math. Phys.* **17** 524
For a review of applications in optics see Dodonov V V 2002 *J. Opt. B: Quantum Semiclass. Opt.* **4** R1
[21] Kuryshkin V V 1980 *Ann. Fond. Louis de Broglie* **5** 111
[22] Drigo Filho E and Candido Ribeiro M A 2000 *Preprint* nucl-th/0108073
[23] Aleixo A N, Balantekin A B and Candido Ribeiro M A 2000 *J. Phys. A: Math. Gen.* **33** 3173 (*Preprint* quant-ph/0001049)
[24] Aleixo A N, Balantekin A B and Candido Ribeiro M A 2001 *J. Phys. A: Math. Gen.* **34** 1109 (*Preprint* quant-ph/0101024)